Oscillation Properties of Solutions for Certain Nonlinear Difference Equations of Third Order

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Abstract : In this paper some sufficient conditions for the oscillation of all solutions of certain difference equations are obtained. Examples are given to illustrate the results.

Key words: Linear, Nonlinear, Difference equations, Oscillations and Non-oscillation.

AMS Subject Classification: 39 A 11.

1 Introduction

We are concerned with the oscillatory properties of all solutions of third order nonlinear difference equations of the form

$$\Delta^2\left(\frac{q_n}{a_n}\Delta\left(x_n+c_nx_{n-\sigma}\right)\right)+p_n\Delta x_n+q_nf(x_{n+1})=0; n=0,1,2,\ldots \quad (1.1)$$

$$\Delta^2\left(\frac{q_n}{a_n}\phi(x_n)\Delta\left(x_n+c_nx_{n-\sigma}\right)\right)+p_n\Delta x_n+q_nf(x_{n+1})=0; n=0,1,2,\ldots \quad (1.2)$$

$$\Delta^2\left(\frac{q_n}{a_n}\Delta\left(x_n+c_nx_{n-\sigma}\right)\right)+q_nf(x_{n+1})=0; n=0,1,2,\ldots \quad (1.3)$$

$$\Delta^2\left(\frac{q_n}{a_n}\phi(x_n)\Delta\left(x_n+c_nx_{n-\sigma}\right)\right)+p_n\Delta x_n+q_nf(x_{n+1})=0; n=0,1,2,\ldots \quad (1.4)$$

Where the following conditions are assumed to hold.

(H1) $\{a_n\},\{p_n\},\{q_n\}$ and $\{c_n\}$ are real positive sequence and $q_n \neq 0$ for infinitely many values of n.

(H2) $f:R\rightarrow R$ is continues and $xf(x)>0$ for all $x\neq 0$.

(H3) there exists a real valued function g such that

$f(u_n)-f(v_n)=g(u_n,v_n)[(u_n+c_nu_{n-\sigma})-(v_n+c_nv_{n-\sigma})]$, for all $u_n\neq 0, v_n\neq 0, c\geq 0, n>\sigma>0$ and

$g(u_n, v_n) \geq L > 0 \in R.$

(H4) $\phi : R \to R$ is continues for all $x \neq 0, \phi(x_n) > 0$.

(H5) $\sum_{n=M}^{\infty} (n+1) p_n^{\;2} < \infty.$

(H6) $\sum_{n=0}^{\infty} \frac{q_n^{\;2}}{a_n^{\;2}} < \infty.$

(H7) $\sum_{n=0}^{\infty} (n+1) q_n = \infty.$

(H8) $\sum_{n=0}^{\infty} \frac{a_n}{n q_n} = \infty.$

By a solution of equation (1.1) –(1.4), we mean a real sequence $\{x_n\}$ satisfying (1.1)-(1.4) for $n = 0,1,2,....$A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called non-oscillatory. The forward difference operator Δ is defined by

$\Delta x_n = x_{n+1} - x_n$

In recent years, much research is going in the study of oscillatory behavior of solutions of third order difference equations. For more details on oscillatory behavior of difference equations, one may refer [1-22].

2 Main Results

In this section, we present some sufficient condition for the oscillation of all the solutions of (1.1)-(1.4). We begin with the following lemma.

Lemma 1

Let $P(n, s, x)$ be defined on $N \times N \times R^+, N = \{0,1,2,...\}, R^+ = [0, \infty)$ such that for fixed n and s, the function $P(n, s, x)$ is non-decreasing in x .Let $\{r_n\}$ be a given sequence and the sequences $\{x_n\}$ and $\{z_n\}$ be defined on N satisfying , for all $n \in N$,

$$x_n \geq r_n + \sum_{s=0}^{n-1} P(n, s, x_s), \tag{2.1}$$

And $$z_n = r_n + \sum_{s=0}^{n-1} P(n,s,z_s), \tag{2.2}$$

respectively. Then $z_n \le x_n$ for all $n \in N$.

This proof can be found in [18].

Theorem 1

In addition to (H1), (H2) and (H3).assume that (H5), (H6), (H7) and (H8) hold and let $z_n = x_n + c_n x_{n-\sigma}$. Then, every solution of (1.1) is oscillatory.

Proof:

Suppose the contrary. Then we may assume that $\{x_n\}$ be a non oscillatory solution of (1.1), such that $x_n > 0 (or x_n < 0)$ for all $n \ge M-1, M > 0$ is an integer and let $b_n = \frac{q_n}{a_n}$.

Equation (1.1) implies

$$\Delta\left(b_{n+1}\Delta z_{n+1}\right) - \Delta\left(b_n \Delta z_n\right) + p_n \Delta x_n + q_n f(x_{n+1}) = 0 \tag{2.3}$$

Multiplying (2.3) by $\frac{n+1}{f(x_{n+1})}$ and summing from M to $(n-1)$, we obtain

$$\sum_{s=M}^{n-1} \frac{s+1}{f(x_{s+1})} \Delta\left(b_{s+1}\Delta z_{s+1}\right) - \sum_{s=M}^{n-1} \frac{s+1}{f(x_{s+1})} \Delta\left(b_s \Delta z_s\right) + \sum_{s=M}^{n-1} \frac{s+1}{f(x_{s+1})} p_s \Delta x_s + \sum_{s=M}^{n-1} (s+1) q_s = 0. \tag{2.4}$$

But

$$\sum_{s=M}^{n-1} \frac{s+1}{f(x_{s+1})} \Delta\left(b_{s+1}\Delta z_{s+1}\right) = \frac{(n+1) b_{n+1} \Delta x_{n+1}}{f(x_{n+1})} - \frac{(M+1) b_{M+1} \Delta x_{M+1}}{f(x_{M+1})} - \sum_{s=M}^{n-1} \frac{b_{s+2} \Delta z_{s+2}}{f(x_{s+2})} + \sum_{s=M}^{n-1} \frac{(s+1) b_{s+2} g(x_{s+2,} x_{s+1}) \Delta z_{s+1} \Delta z_{s+2}}{f(x_{s+1}) f(x_{s+2})} \tag{2.5}$$

Also,

$$\sum_{s=M}^{n-1} \frac{s+1}{f(x_{s+1})} \Delta\left(b_s \Delta z_s\right) = \frac{(n+1) b_n \Delta x_n}{f(x_{n+1})} - \frac{(M+1) b_M \Delta x_M}{f(x_{M+1})} - \sum_{s=M}^{n-1} \frac{b_{s+1} \Delta z_{s+1}}{f(x_{s+2})} + \sum_{s=M}^{n-1} \frac{(s+1) b_{s+1} g(x_{s+2,} x_{s+1}) (\Delta z_{s+1})^2}{f(x_{s+1}) f(x_{s+2})} \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.4), we have

$$\left(\frac{(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})}-\frac{(n+1)b_n\Delta z_n}{f(x_{n+1})}\right)+\sum_{s=M}^{n-1}\left(\frac{(s+1)b_{s+2}g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}-\frac{(s+1)b_{s+1}g(x_{s+2},x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})}\right)$$

$$-\sum_{s=M}^{n-1}\left(\frac{b_{s+2}\Delta z_{s+2}}{f(x_{s+2})}-\frac{b_{s+1}\Delta z_{s+1}}{f(x_{s+2})}\right)+\sum_{s=M}^{n-1}\frac{s+1}{f(x_{s+1})}p_s\Delta x_s+\sum_{s=M}^{n-1}(s+1)q_s=\left(\frac{(M+1)b_{M+1}\Delta z_{M+1}}{f(x_{M+1})}-\frac{(M+1)b_M\Delta z_M}{f(x_{M+1})}\right) \quad (2.7)$$

Using Schwarz's inequality, we have

$$\sum_{s=M}^{n-1}\left(\frac{b_{s+2}\Delta z_{s+2}}{f(x_{s+2})}\right)\le\left(\sum_{s=M}^{n-1}(b_{s+2})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{\Delta z_{s+2}}{f(x_{s+2})}\right)^2\right)^{1/2} \quad (2.8)$$

$$\sum_{s=M}^{n-1}\left(\frac{b_{s+1}\Delta z_{s+1}}{f(x_{s+2})}\right)\le\left(\sum_{s=M}^{n-1}(b_{s+1})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{\Delta z_{s+1}}{f(x_{s+2})}\right)^2\right)^{1/2} \quad (2.9)$$

$$\sum_{s=M}^{n-1}\left(\frac{(s+1)b_{s+2}g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}\right)\le\left(\sum_{s=M}^{n-1}(b_{s+2})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{(s+1)g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}\right)^2\right)^{1/2} \quad (2.10)$$

$$\sum_{s=M}^{n-1}\left(\frac{(s+1)b_{s+1}g(x_{s+2},x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})}\right)\le\left(\sum_{s=M}^{n-1}(b_{s+1})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{(s+1)g(x_{s+2},x_{s+1})(\Delta z_{s+1})^4}{f(x_{s+1})f(x_{s+2})}\right)^2\right)^{1/2} \quad (2.11)$$

And

$$\sum_{s=M}^{n-1}\left(\frac{(s+1)p_s\Delta x_s}{f(x_{s+1})}\right)\le\left(\sum_{s=M}^{n-1}(s+1)(p_s)^2\right)^{1/2}\left(\sum_{s=M}^{n-1}(s+1)\left(\frac{\Delta x_s}{f(x_{s+1})}\right)^2\right)^{1/2} \quad (2.12)$$

In view of (2.8), (2.9), (2.10),(2.11) and (2.12), the summation in (2.7) is bounded , we have

$$\left(\frac{(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})}-\frac{(n+1)b_n\Delta z_n}{f(x_{n+1})}\right)-\left(\sum_{s=M}^{n-1}(b_{s+2})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{\Delta z_{s+2}}{f(x_{s+2})}\right)^2\right)^{1/2}+\left(\sum_{s=M}^{n-1}(b_{s+1})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{\Delta z_{s+1}}{f(x_{s+2})}\right)^2\right)^{1/2}$$

$$+\left(\sum_{s=M}^{n-1}(b_{s+2})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{(s+1)g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}\right)^2\right)^{1/2}-\left(\sum_{s=M}^{n-1}(b_{s+1})^2\right)^{1/2}\left(\sum_{s=M}^{n-1}\left(\frac{(s+1)g(x_{s+2},x_{s+1})(\Delta z_{s+1})^4}{f(x_{s+1})f(x_{s+2})}\right)^2\right)^{1/2}$$

$$+\left(\sum_{s=M}^{n-1}(s+1)(p_s)^2\right)^{1/2}\left(\sum_{s=M}^{n-1}(s+1)\left(\frac{\Delta x_s}{f(x_{s+1})}\right)^2\right)^{1/2}\le\left(\frac{(M+1)b_{M+1}\Delta z_{M+1}}{f(x_{M+1})}-\frac{(M+1)b_M\Delta z_M}{f(x_{M+1})}\right)-\sum_{s=M}^{n-1}(s+1)q_s \quad (2.13)$$

In view of (H5), (H6) and (H7), we get from (2.13) that $\dfrac{(n+1)\Delta(b_n\Delta z_n)}{f(x_{n+1})} \to -\infty$ as $n \to \infty$.

Hence there exists $M_1 \geq M$ such that $\Delta(b_n\Delta z_n) < 0$ for $n \geq M$, which implies $\Delta(b_n\Delta z_n) < -k, k > 0$

Summing the last inequality from m to $(n-1)$, we obtain

$$\sum_{s=m}^{n-1}\Delta(b_s\Delta z_s) < \sum_{s=m}^{n-1}(-k)$$

$$\text{That is } b_n\Delta z_n < -k(n-m) + b_m\Delta z_m$$

Therefore $b_n\Delta z_n \to -\infty$ as $n \to \infty$. Hence there exists $M_2 \geq M_1$ such that $\Delta z_n < 0$ for $n \geq M_2$ (2.14)

Rewriting (2.7), we have

$$\frac{(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})} + \sum_{s=M_2}^{n-1}\frac{(s+1)b_{s+2}g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})} = \frac{(n+1)b_n\Delta z_n}{f(x_{n+1})} + \frac{(M+1)b_{M+1}\Delta z_{M+1}}{f(x_{M+1})} - \frac{(M+1)b_M\Delta z_M}{f(x_{M+1})}$$

$$-\sum_{s=M}^{n-1}(s+1)q_s + \sum_{s=M_2}^{n-1}\frac{(s+1)b_{s+1}g(x_{s+2},x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})} - \sum_{s=M}^{M_2-1}\frac{(s+1)b_{s+2}g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})} - \sum_{s=M}^{M_2-1}\frac{s+1}{f(x_{s+1})}p_s\Delta x_s$$

$$+\sum_{s=M}^{M_2-1}\frac{(s+1)b_{s+1}g(x_{s+2},x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})} + \sum_{s=M}^{M_2-1}\left(\frac{b_{s+2}\Delta z_{s+2}}{f(x_{s+2})} - \frac{b_{s+1}\Delta z_{s+1}}{f(x_{s+2})}\right) + \sum_{s=M_2}^{n-1}\left(\frac{b_{s+2}\Delta z_{s+2}}{f(x_{s+2})} - \frac{b_{s+1}\Delta z_{s+1}}{f(x_{s+2})}\right) - \sum_{s=M_2}^{n-1}\frac{s+1}{f(x_{s+1})}p_s\Delta x_s$$

(2.15)

From (H1), (H7), (2.14) and (2.15), there exists an integer $M_3 \geq M_2$, such that

$$\frac{(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})} + \sum_{s=M_2}^{n-1}\frac{(s+1)b_{s+2}g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})} \leq -l, l \geq M_3 \text{ where } l \text{ is a positive integer.}$$

$$-\frac{(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})} - \sum_{s=M_2}^{n-1}\frac{(s+1)b_{s+2}g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})} \geq l \qquad (2.16)$$

Let $u_{n+1} = -(n+1)\Delta z_{n+1}$, (2.16) becomes

$$\frac{u_{n+1}b_{n+1}}{f(x_{n+1})} \geq l + \sum_{s=M_3}^{n-1}\frac{(s+1)b_{s+2}g(x_{s+2},x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}; n \geq M_3$$

(i.e) $$u_{n+1} \geq l\frac{f(x_{n+1})}{b_{n+1}} + \sum_{s=M_3}^{n-1} \frac{b_{s+2} f(x_{n+1}) g(x_{s+2}, x_{s+1})(-\Delta z_{s+2}) u_{s+1}}{b_{n+1} f(x_{s+1}) f(x_{s+2})} \quad (2.17)$$

Also, Let $$v_{n+1} = l\frac{f(x_{n+1})}{b_{n+1}} + \sum_{s=M_3}^{n-1} \frac{b_{s+2} f(x_{n+1}) g(x_{s+2}, x_{s+1})(-\Delta z_{s+2}) v_{s+1}}{b_{n+1} f(x_{s+1}) f(x_{s+2})} \quad (2.18)$$

Using lemma 1, we have, from (2.17) and (2.18)

$$\Rightarrow u_{n+1} \geq v_{n+1} \quad (2.19)$$

(2.18) implies $$v_{n+1} = \frac{f(x_{n+1})}{b_{n+1}} \left(l + \sum_{s=M_3}^{n-1} \frac{b_{s+2} g(x_{s+2}, x_{s+1})(-\Delta z_{s+2}) v_{s+1}}{f(x_{s+1}) f(x_{s+2})} \right)$$

This implies that $$v_{n+1} \geq \frac{lf(x_{M_3})}{b_{n+1}}; n \geq M_3 \quad (2.20)$$

From (2.19) and (2.20), we have $-(n+1)\Delta z_{n+1} \geq \dfrac{lf(x_{M_3})}{b_{n+1}}$

$$\Rightarrow \Delta z_{n+1} \leq \frac{-lf(x_{M_3})}{(n+1)b_{n+1}} \quad (2.21)$$

Summing (2.21) from M_3 to $(n-1)$, we have $\displaystyle\sum_{s=M_3}^{n-1} \Delta z_{n+1} \leq -lf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(n+1)b_{n+1}}$

That is $$z_{n+1} - z_{M_3+1} \leq -lf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(n+1)b_{n+1}}$$

$$\Rightarrow z_{n+1} \leq z_{M_3+1} - lf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(n+1)b_{n+1}}$$

$$\Rightarrow z_n = (x_n + c_n x_{n-\sigma}) \leq 0 \text{ For sufficiently large } n,$$

Which is a contradiction to the fact that x_n is eventually positive. The proof is similar for the case when x_n is eventually negative. Hence the theorem is completely proved.

Examples

Example 1

Consider the difference equation

$$\Delta^2\left(\frac{n}{n+1}\Delta\left(x_n+nx_{n-3}\right)\right)+\frac{9n^2+18n+5}{2n^2(n+1)(n+2)}\Delta x_n+\frac{x_{n+1}}{n(n+1)}=0 \qquad \text{(E1)}$$

All the conditions of Theorem 1 are satisfied. Hence every solution of equation (E1) is oscillatory.

Example 2

Consider the difference equation

$$\Delta^2\left(\frac{n+1}{n+2}\Delta\left(x_n+nx_{n-5}\right)\right)+\frac{1}{n^3}\sqrt{\frac{n}{n+1}}\Delta x_n+\frac{(x_{n+1})^3}{(n+1)(n+2)}=0 \qquad \text{(E2)}$$

All the conditions of Theorem 1 are satisfied. Hence every solution of equation (E2) is oscillatory

Theorem 2

In addition to (H1), (H2) ,(H3)and (H4).assume that (H5), (H6), (H7) and (H8) hold and let $z_n = x_n + c_n x_{n-\sigma}$. Then, every solution of (1.2) is oscillatory.

Theorem 3

In addition to (H1), (H2) and (H3).assume that (H6), (H7) and (H8) hold and let $z_n = x_n + c_n x_{n-\sigma}$. Then, every solution of (1.3) is oscillatory.

Theorem 4

In addition to (H1), (H2), (H3) and (H4).assume that (H6), (H7) and (H8) hold and let $z_n = x_n + c_n x_{n-\sigma}$. Then, every solution of (1.4) is oscillatory.

Proofs of Theorem 2, Theorem 3 and Theorem 4 are similar to the proof of Theorem 1 and hence the details are omitted.

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